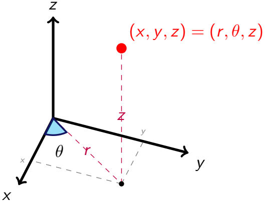
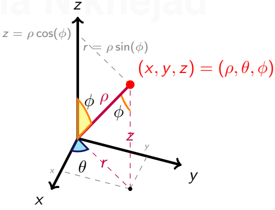
Section 15.4

Integration in Polar, Cylindrical, and Spherical Coordinates

Cylindrical Coordinates



Spherical Coordinates



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PreLecture Video

Double Integrals in Polar Coordinates

Recall the **change-of-variables formula for double integrals:** if $G(u, v) = (x(u, v), y(u, v))$ is a transformation with $G(S) = R$, then

$$\iint_R f(x, y)\, dA_{xy} = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA_{uv}.$$

The transformation from polar to rectangular coordinates is $G(r, \theta) = (x, y) = (r\cos(\theta), r\sin(\theta))$, with Jacobian $\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = r$.

Double Integral Formula in Polar Coordinates

$$\iint_R f(x, y)\, dA_{xy} = \iint_S f(r\cos(\theta), r\sin(\theta))\, \boxed{r}\, dA_{r\theta}$$

where the region R is described in rectangular coordinates x, y on the left side, and in polar coordinates r, θ on the right side.

1 Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates

The double integral formula in polar coordinates is

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

or more simply

$$\iint_R f\, dx\, dy = \iint_S f\, r\, dr\, d\theta$$

or even more simply

$$dA \;=\; dx\, dy \;=\; r\, dr\, d\theta.$$

The object dA is sometimes called the **area element.** This is useful for regions that are easier to describe in polar coordinates.

Example 1: Let D be the disk of radius 1 centered at $(0,0)$. Evaluate

$$\iint_D e^{x^2+y^2}\, dA\,.$$

<u>Solution:</u> In polar coordinates,

$$D = \left\{(r,\theta) \;\middle|\; 0 \le r \le 1,\ 0 \le \theta < 2\pi\right\}.$$

Since $r^2 = x^2 + y^2$,

$$\iint_D e^{x^2+y^2}\, dA \;=\; \int_0^{2\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta \;=\; \int_0^{2\pi} \left[\frac{e^{r^2}}{2}\right]_0^1 d\theta$$

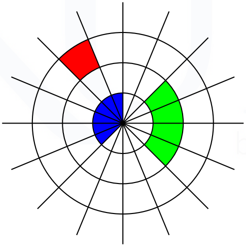
$$= \int_0^{2\pi} \frac{1}{2}(e-1)\, d\theta \qquad = \pi(e-1)$$

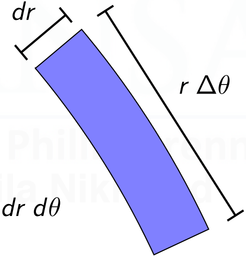
Polar Rectangles

In general, a region described in polar coordinates by

$$R = \{(r, \theta) : \; a \leq r \leq b, \; p \leq \theta \leq q\}$$

is called a **polar rectangle**. It corresponds to a section of an annulus:





$$dA = r \, dr \, d\theta$$

(Recall that the measure of an angle in radians is $\dfrac{\text{arc length}}{\text{radius}}$.)

2 Triple Integrals in Cylindrical and Spherical Coordinates Formulas

Triple Integrals in Cylindrical and Spherical Coordinates

Remember the change-of-variables formula for triple integrals:

Let $G(u,v,w) = (x,y,z)$ be a transformation with $G(S) = R$. Then

$$\iiint_R f(x,y,z)\,dV_{xyz} = \iiint_S f(G(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV_{uvw}.$$

Transformations from cylindrical or spherical to rectangular coordinates:

$$G(r,\theta,z) = (r\cos(\theta),\ r\sin(\theta),\ z) \qquad \text{(cylindrical)}$$

$$H(\rho,\phi,\theta) = (\rho\sin(\phi)\cos(\theta),\ \rho\sin(\phi)\sin(\theta),\ \rho\cos(\phi)) \qquad \text{(spherical)}$$

$$\text{Jac}(G) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{r}$$

$$\text{Jac}(H) = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{vmatrix} = \boxed{\rho^2\sin(\phi)}$$

Triple Integrals in Cylindrical Coordinates

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{G^{-1}(R)} f(G(r,\theta,z))\,\boxed{r}\,dr\,d\theta\,dz$$

Triple Integrals in Spherical Coordinates

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{H^{-1}(R)} f(H(\rho,\phi,\theta))\,\boxed{\rho^2\sin(\phi)}\,d\rho\,d\phi\,d\theta$$

where $G(r,\theta,z) = (x,y,z)$ and $H(\rho,\phi,\theta) = (x,y,z)$.

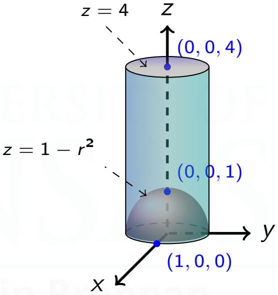
Or, for short:

Volume Element in $\mathbb{R}^3$

$$dV \;=\; dx\,dy\,dz \;=\; r\,dr\,d\theta\,dz \;=\; \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta.$$

Cylindrical Triple Integrals: Example

Example 2: Integrate $f(x, y, z) = x^2 + y^2$ over the solid S contained inside the cylinder $x^2 + y^2 = 1$, under the plane $z = 4$, and above the elliptic paraboloid $z = 1 - x^2 - y^2$.



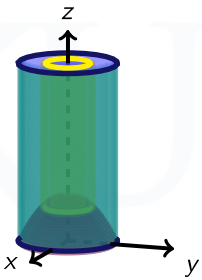
Solution: In cylindrical coordinates, the solid S has bounds $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$, $1 - r^2 \leq z \leq 4$.

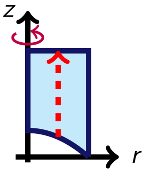
$$\iiint_S (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 r^3 \, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 3r^3 + r^5 \, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^4}{4} + \frac{r^6}{6} \right]_0^1 d\theta = \int_0^{2\pi} \frac{11}{12}\, d\theta = \boxed{\frac{11\pi}{6}}$$

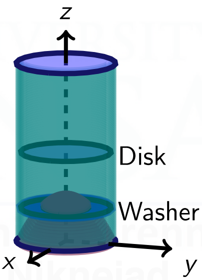
Cylindrical Integrals: Example 2 Continued

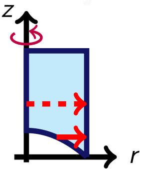
The cylindrical integral set-up in order $dz\,dr\,d\theta$ can be described similar to the shell method.





The cylindrical integral set-up in order $dr\,dz\,d\theta$, can be described similar to the washer method.





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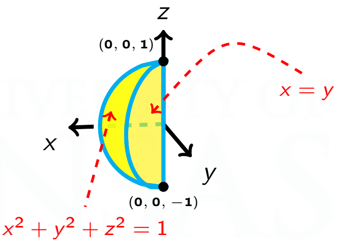
Spherical Triple Integrals: Example

Example 3: Set up (but do not evaluate) the integral $\iiint_W y\,dv$, where W is the "orange slice" bounded by the unit ball and the planes $y = 0$ and $y = x$, as shown.



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Solution: In spherical coordinates,

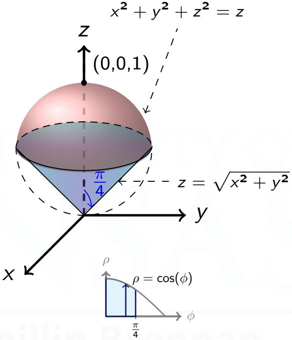
$$W = \{(\rho,\,\phi,\,\theta) \mid 0 \le \rho \le 1,\ 0 \le \phi \le \pi,\ 0 \le \theta \le \pi/4\}$$

so

$$\iiint_W y\,dv = \int_0^1 \int_0^\pi \int_0^{\pi/4} \underbrace{\rho \sin\phi \sin\theta}_{y}\ \underbrace{\rho^2 \sin(\phi)\,d\theta\,d\phi\,d\rho}_{dV}$$

Spherical Triple Integrals: Example

Example 4: Find the volume of the solid contained above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.



Solution: In spherical coordinates, the solid has bounds $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \rho \le \cos(\phi)$.

(For the bound on ρ, write $x^2 + y^2 + z^2 = z$ in spherical coordinates as $\rho^2 = \rho\cos(\phi)$.)

$$\iiint_S 1\, dV = \int_0^{\frac{\pi}{4}} \int_0^{\cos(\phi)} \int_0^{2\pi} \rho^2 \sin(\phi)\, d\theta\, d\rho\, d\phi$$

$$= \frac{2\pi}{3} \underbrace{\int_0^{\frac{\pi}{4}} \sin(\phi)\cos^3(\phi)\, d\phi}_{u=\cos(\phi),\ du=-\sin(\phi)d\phi} = \frac{2\pi}{3} \int_1^{\frac{\sqrt{2}}{2}} -u^3\, du = \frac{\pi}{8}$$